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LETTER TO THE EDITOR

q-Laguerre and Wall polynomials are related by the Fourier–Gauss transform

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Abstract. It is shown that the q-Laguerre and Wall (or little q-Laguerre) polynomials are interrelated by the Fourier–Gauss transform. In the limit when the degree of these polynomials tends to infinity, this integral transform provides the relation between Jackson's second and third q-Bessel functions.

The q-Laguerre polynomials $L_n^{(\alpha)}(x;q)$ are defined [1–3] as

$$L_n^{(\alpha)}(x;q) := \frac{(q^{\alpha+1};q)_n}{(q;q)_n}\,{}_1\phi_1(q^{-n};q^{\alpha+1};q,-xq^{n+\alpha+1})$$
$$= \frac{(q^{\alpha+1};q)_n}{(q;q)_n}\sum_{k=0}^{n}\frac{(q^{-n};q)_k\,x^k}{(q^{\alpha+1},q;q)_k}\,q^{k(n+\alpha)+k(k+1)/2} \tag{1}$$

where we have used the standard notation for the q-shifted factorial $(z;q)_n$ and the basic hypergeometric series ${}_1\phi_1(a;b;q,z)$ (see, for example, [4] or [5]). These polynomials can be generated by the three-term recurrence relation

$$-q^{2n+\alpha+1}xL_n^{(\alpha)}(x;q) = (1-q^{n+1})\,L_{n+1}^{(\alpha)}(x;q) - [(1-q^{n+1})+q(1-q^{n+\alpha})]\,L_n^{(\alpha)}(x;q)$$
$$+q(1-q^{n+\alpha})\,L_{n-1}^{(\alpha)}(x;q) \tag{2}$$

with the initial conditions $L_{-1}^{(\alpha)}(x;q) = 0$ and $L_0^{(\alpha)}(x;q) = 1$. In the limit when the parameter q tends to 1, they converge to the ordinary Laguerre polynomials $L_n^{(\alpha)}(x)$, namely

$$\lim_{q\to 1} L_n^{(\alpha)}((1-q)x;q) = L_n^{(\alpha)}(x)\,. \tag{3}$$

Since the Stieltjes and Hamburger moment problems associated with the q-Laguerre polynomials are indeterminate, the measure with respect to which they are orthogonal is not unique [3, 6–8]. Examples of absolutely continuous and discrete orthogonality relations for $L_n^{(\alpha)}(x;q)$ have been given by Moak [3]. He has also proved that the q-Laguerre polynomials converge to the entire function

$$\lim_{n\to\infty} L_n^{(\alpha)}(x;q) = x^{-\alpha/2}\,J_\alpha^{(2)}(2\sqrt{x};q) \tag{4}$$

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the roots of which are the mass points of an extreme measure. Jackson's q-Bessel function $J_\alpha^{(2)}(z;q)$ in (4) is defined [9, 10] as

$$J_\alpha^{(2)}(z;q) := \frac{1}{(q;q)_\alpha} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+\alpha)}}{(q^{\alpha+1},q;q)_n} \left(\frac{z}{2}\right)^{2n+\alpha} . \qquad (5)$$

Applying the inversion formula (with respect to the change $q \to q^{-1}$)

$$(z;q^{-1})_n = q^{-n(n-1)/2}(-z)^n\,(z^{-1};q)_n \qquad (6)$$

to the definition (1) gives the relation

$$\begin{aligned} L_n^{(\alpha)}(x;q^{-1}) &= q^{-n\alpha} \frac{(q^{\alpha+1};q)_n}{(q;q)_n} \sum_{k=0}^{n} \frac{(q^{-n};q)_k}{(q^{\alpha+1},q;q)_k} (-qx)^k \\ &= q^{-n\alpha} \frac{(q^{\alpha+1};q)_n}{(q;q)_n}\, p_n(-x;q^\alpha|q) \end{aligned} \qquad (7)$$

between the q-Laguerre and Wall (or little q-Laguerre) polynomials $p_n(x;a|q)$ [5]. The latter polynomials are defined [11] as

$$p_n(x;a|q) := {}_2\phi_1(q^{-n},0;aq;q,qx) = \sum_{k=0}^{n} \frac{(q^{-n};q)_k\,(qx)^k}{(aq,q;q)_k} . \qquad (8)$$

In exactly the same way as for (4), one can prove that the polynomials (8) also converge to the entire function

$$\lim_{n\to\infty} p_n(q^n x;q^\alpha|q) = \frac{(q;q)_\alpha}{x^{\alpha/2}}\, J_\alpha^{(3)}(\sqrt{x};q) \qquad (9)$$

where Jackson's third q-Bessel function $J_\alpha^{(3)}(z;q)$ is given [12, 13] by the relation

$$J_\alpha^{(3)}(z;q) := \frac{1}{(q;q)_\alpha} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(q^{\alpha+1},q;q)_n}\, z^{2n+\alpha} . \qquad (10)$$

This short letter claims to prove that the polynomials (1) and (8) are in fact related to each other by the classical Fourier–Gauss transform. As the degree of both polynomials tends to infinity, this integral transform in turn yields the corresponding relation between the q-Bessel functions (5) and (10).

To that end, let us denote $q = \exp(-2\kappa^2)$ and evaluate an integral

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} L_n^{(\alpha)}(t\mathrm{e}^{2\kappa s};q)\,\exp(2\mathrm{i}rs - s^2)\,\mathrm{d}s \qquad (11)$$

where t is a constant. Substitute the finite sum (1) for $L_n^{(\alpha)}(x;q)$ with respect to the variable $x = t\mathrm{e}^{2\kappa s}$ in (11) and use the well-known Fourier transform

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(2\mathrm{i}rs - s^2)\,\mathrm{d}s = \exp(-r^2) \qquad (12)$$

for the Gauss exponential function $\exp(-s^2)$. This gives

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} L_n^{(\alpha)}(t\mathrm{e}^{2\kappa s};q)\,\exp(2\mathrm{i}rs - s^2)\,\mathrm{d}s = q^{n\alpha}\, L_n^{(\alpha)}(-q^{n+\alpha-1/2}\,t\mathrm{e}^{2\mathrm{i}\kappa r};q^{-1})\,\exp(-r^2) . \qquad (13)$$

Taking equation (7) into account, this result can be written in the equivalent form

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} L_n^{(\alpha)}(t\mathrm{e}^{2\kappa s}; q)\ \exp(2\mathrm{i}rs - s^2)\ \mathrm{d}s$$
$$= \frac{(q^{\alpha+1}; q)_n}{(q; q)_n}\ p_n(tq^{n+\alpha-1/2}\,\mathrm{e}^{2\mathrm{i}\kappa r}; q^\alpha|q)\ \exp(-r^2)\,. \tag{14}$$

Note that since the q-Charlier polynomials $C_n(x; -q^{-\alpha}; q) = (q; q)_n\, L_n^{(\alpha)}(-x; q)$, one may regard (14) as a Fourier–Gauss transform between the q-Charlier and Wall polynomials (8).

In view of the limiting relations (4) and (9) we may let $n \to \infty$ on both sides of (14). After some simple manipulation, this results in the Fourier–Gauss transform

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} J_\alpha^{(2)}(2t\mathrm{e}^{\kappa s}; q)\ \exp(2\mathrm{i}rs - s^2)\ \mathrm{d}s = q^{\alpha(2-3\alpha)/8}\, J_\alpha^{(3)}(q^{(\alpha-1)/4}t\mathrm{e}^{\mathrm{i}\kappa r}; q)\ \exp(-r^2) \tag{15}$$

relating Jackson's second and third q-Bessel functions (5) and (10), respectively. Integral transforms of this type for Jackson's q-Bessel functions $J_\alpha^{(i)}(z; q)$, $i = 1, 2, 3$, have recently been discussed in [14]. It has been proved in particular that (see [14, equation (21)])

$$J_\alpha^{(3)}(q^{(\alpha-1)/4}t\mathrm{e}^{-\kappa x}; q)\ \exp(-x^2) = q^{\alpha(\alpha-2)/8} \int_{-\infty}^{\infty} \frac{\mathrm{d}y}{\sqrt{\pi}}\, J_\alpha^{(1)}(2t\mathrm{e}^{\mathrm{i}\kappa y}; q)\ \exp\left(2\mathrm{i}xy - y^2\right) \tag{16}$$

where [9, 10]

$$J_\alpha^{(1)}(z; q) := \frac{1}{(q; q)_\alpha} \sum_{n=0}^{\infty} \frac{(-1)^n\ (z/2)^{2n+\alpha}}{(q^{\alpha+1}, q; q)_n}\,. \tag{17}$$

By the aid of the inversion formula (6) it is not hard to verify that the Fourier–Gauss transforms (15) and (16) are interrelated by a replacement of the base $q \to q^{-1}$.

Our avowed interest in the classical Fourier integral transform is in its usefulness as a tool in revealing close relations between various q-special functions. Some other instances of such remarkable pertinence of the Fourier transformations have been already discussed in [14–16]. We believe that further study in this direction will help to fathom the properties of q-special functions.

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